Numerical Simulation of Relativistic Astrophysical Flows

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Magnetoplasmic processes in relativistic astrophysics

Outline of the lecture

1. The equations of relativistic fluid dynamics
2. Discretization of the equations of relativistic fluid dynamics
3. Code/method validation
4. Applications
5. Summary

Contents

The equations of GRHD

The equations describing the evolution of a relativistic fluid are *conservation laws*:

$$\nabla \cdot \mathbf{J} = 0, \qquad \text{Conservation of the baryon number}$$

$$\nabla \cdot \mathbf{T} = 0, \qquad \text{Conservation of the energy-momentum}$$

$\nabla \equiv$ covariant divergence, $\mathbf{J} \equiv$ current of rest mass, $\mathbf{T} \equiv$ energy-momentum tensor.

For a perfect fluid (=shear or heat conduction) and using $G = c = 1$

$$J^\mu = \rho u^\mu, \qquad T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$\rho \equiv$ rest-mass density,
$p \equiv$ pressure,
$h = 1 + \epsilon + p/\rho \equiv$ specific enthalpy,
$\epsilon \equiv$ specific internal energy,
$u^\mu \equiv$ 4-velocity of the fluid ($u^\mu u_\mu = -1$),
$g_{\mu\nu} \equiv$ metric of the spacetime $\mathcal{M}$.

The systems of equations is closed with an equation of state (EoS), for instance, $p = p(\rho, \epsilon, \ldots)$.

Approximations

Special Relativity (RHD)	$g_{\mu\nu} = \eta_{\mu\nu}$ gravitational field neglected	- Rel. heavy-ion collisions - Extragalactic jets - GRB afterglows
External field (GRHD)	$g_{\mu\nu} \neq f(t)$ background metric	- Accretion onto compact objects - jet formation - proto-GRBs
Dynamical $\mathcal{M}$ (GRHD + EE)	$g_{\mu\nu} = f(t)$ $g_{\mu\nu}$ from EE	- GR stellar core collapse - NS, BH mergers

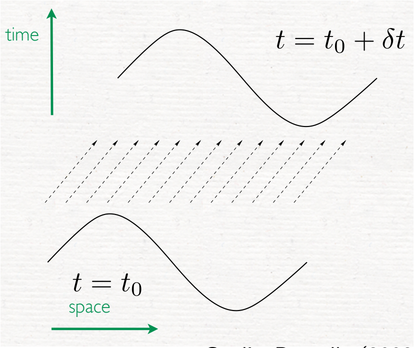
Simple examples: (i) advection equation

Before looking at the solution of the hydrodynamical equations there are some fundamental aspects of their nonlinear properties which can be more easily understood considering simple examples.

The simplest linear hyperbolic equation is the advection equation

$$\partial_t U(t,x) + \partial_x U(t,x) = 0$$

The solution is the initial data simply translated in space and time. The propagation speeds are constant everywhere (linear nature of the equation)



Credit: Rezzolla (2008)

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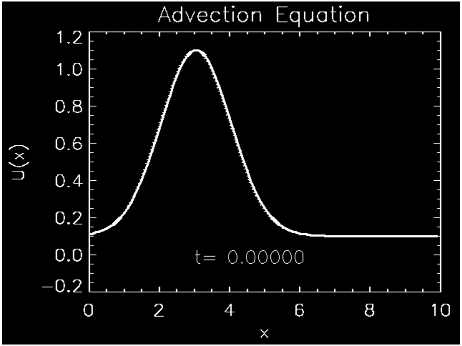

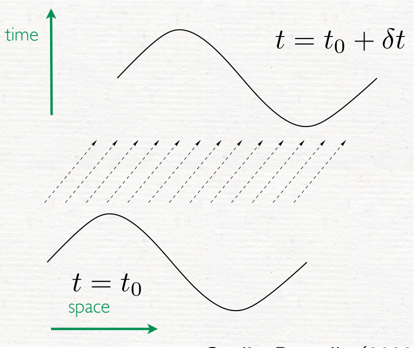

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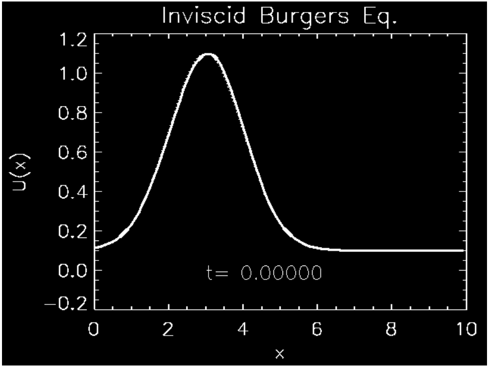
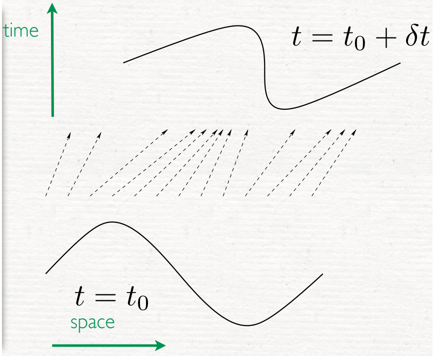
Simple examples: (ii) Burgers' equation

The simplest nonlinear hyperbolic equation is the Burgers' equation

$$\partial_t U(t,x) + U(t,x)\partial_x U(t,x) = \epsilon \partial^2_{xx} U(t,x)$$

where $\epsilon \to 0$ in the inviscid limit. The solution to this eq. is very different because of the dependence of the advection velocity ($U(t,x)$) on t and x.





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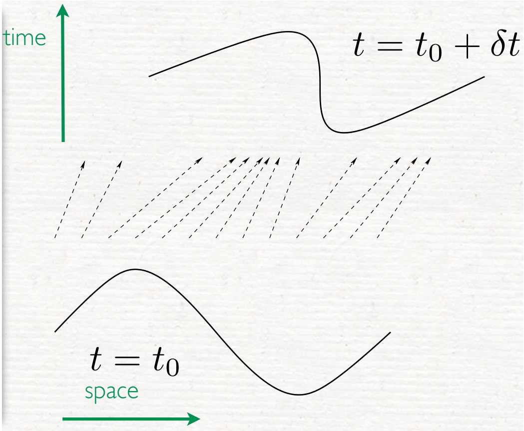
Simple examples: (ii) Burgers' equation

This behaviour is referred to as "shock steepening" and is the consequence that the propagation speeds are not constant as for the advection equation but are a function of space and time (nonlinear nature of the equation).

Stated differently, the maxima of the waves move "faster" than the minima and tend to "catch-up".

1. This is a property of the equations and not of the initial data. Even smooth initial data will (eventually) shock in inviscid fluids.
2. **Numerical challenge:** Once a shock forms we cannot *simply* translate the initial data forward in time from the (given) initial data.



Credit: Rezzolla (2008)

Conservative form of the equations

The homogeneous partial differential equation

$$\partial_t U(t,x) + a(U(t,x))\partial_x U(t,x) = 0$$

is said to be in flux-conservative (FC) form if written as

$$\partial_t U(t,x) + \partial_x F(U(t,x)) = 0$$

Theorems (Lax, Wendroff; Hou, LeFloch)

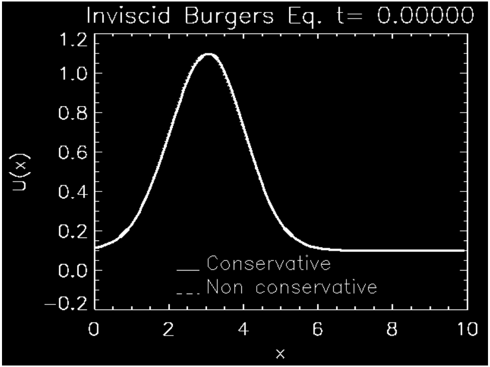
- FC formulation converges to the weak solution of the problem (i.e., a solution of the integral form of the FC form).
- NFC converges to the wrong weak solution of the problem.

In conservative systems (e.g., the hydrodynamic eqs) one usually deals with **a set of equations in FC form**. Hence, the function U and the flux $F(U)$ are replaced by a state vector $\mathbf{U}$ and a flux vector $\mathbf{F}(\mathbf{U})$.

Why do we care at all about the FC/NFC form of the equations?...

Conservation vs non-conservation

Burgers' inviscid equation with continuous initial data offers a good example of the importance of a conservative writing of the equations.



Consider $\partial_t U + U \partial_x U = 0$ with

$$U(0,x) = 0.1 + \exp\left(-\frac{(x-3)^2}{2}\right)$$

The equation can then be written as (dashed line)

$$\partial_t U + U \partial_x U = 0$$

or as (solid line):

$$\partial_t U + \frac{1}{2}\partial_x U^2 = 0$$

Mathematically equivalent but the numerical difference is obvious (dashed line moves at a wrong propagation speed!)

Conservation vs non-conservation: linear systems

For linear systems of eqs, the importance of a conservative formulation is clear as it allows for analytic solutions.
Rewrite the flux conservative equations

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \quad \Longleftrightarrow \quad \partial_t \mathbf{U} + \mathcal{B}\partial_x \mathbf{U} = 0, \tag{1}$$

where $\mathcal{B}(\mathbf{U}) \equiv \partial \mathbf{F}(\mathbf{U})$ is the Jacobian matrix of *constant* coefficients (because the problem is linear).

We next diagonalize $\mathcal{B}(\mathbf{U})$ so that $\Lambda = \mathbf{R}^{-1}\mathcal{B}\,\mathbf{R}$ is the diagonal matrix of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the N linear equations [$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$]

The columns of the matrix $\mathbf{R}$, $\mathbf{R}^i$ are the set of right eigenvectors of $\mathcal{B}$.

Let's recall that the diagonalization and full spectral decomposition are guaranteed to be possible IF we deal with a set of hyperbolic equations. Indeed, the set of eqs

$$\partial_t \mathbf{U} + \mathcal{B}\partial_x \mathbf{U} = 0,$$

is said to be (strongly) hyperbolic iff $\mathcal{B}$ is diagonalizable with a set of real (distinct) eigenvalues λ_i and correspondingly a set of linearly independent (right) eigenvectors $\mathbf{R}^i$.

Characteristic curves

We can now define the **characteristic variables**

$$\mathcal{U} \equiv \mathbf{R}^{-1}\mathbf{U}$$

so that system (1) can be written as

$$\partial_t \mathcal{U} + \Lambda \partial_x \mathcal{U} = 0 \qquad (2)$$

Since Λ is diagonal, Eq. (2) corresponds to a system of N decoupled PDEs.

If $\mathcal{U}^i$ is the i-component of the vector $\mathcal{U}$, we have

$$\partial_t \mathcal{U}^i + \lambda_i \partial_x \mathcal{U}^i = 0 \iff \frac{\mathrm{d}\mathcal{U}^i}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\partial x}{\partial t} = \lambda_i(\mathbf{U}(t,x)) \qquad (3)$$

so that the characteristic variables are constant along those **characteristic curves** in the (x, t) plane having slope λ_i, also known as **characteristic speeds**.

As they remain constant along characteristics, the value the characteristic variables at any time is known once the initial one is determined, i.e.

$$\mathcal{U}^i(t,x) = \mathcal{U}^i(t=0, x - \lambda_i t) \qquad (4)$$

Characteristic *solution*

From the solution in characteristic variables (Eq. 4) we can go back to the original estate vector

$$\mathcal{U} \equiv \mathbf{R}^{-1}\mathbf{U} \quad \Longrightarrow \quad \mathbf{U} = \mathbf{R}\mathcal{U}$$

so that

$$\mathbf{U}(t,x) = \sum_{i=1}^{N} \mathcal{U}^i(t,x)\mathbf{R}^i = \sum_{i=1}^{N} \mathcal{U}^i(0, x - \lambda_i t)\mathbf{R}^i \tag{5}$$

What Eq. (5) expresses is that the solution at any time can be seen as the linear superposition of N waves, each propagating independently at the speed given by the corresponding eigenvalue.

▷ Solutions are built along the time by solving initial value problems (IVP).

Equations of Special Relativistic Hydrodynamics (I)

In Cartesian coordinates, $x^\mu = (t, x, y, z)$, $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ (*e.g.*, Font *et al.* 1994):

$$\begin{aligned} J^\mu{}_{,\mu} &= 0, \\ T^{\mu\nu}{}_{,\mu} &= 0 \qquad (_{,\mu} \equiv \frac{\partial}{\partial x^\mu}) \end{aligned}$$

From the normalization condition $u^\mu u_\mu = -1$:

$$u^\mu = \Gamma(1, v^x, v^y, v^z) \tag{6}$$

$$\Gamma = (1 - v^2)^{-1/2} \quad ; \; v^2 = (v^x)^2 + (v^y)^2 + (v^z)^2.$$

Fundamental system of equations in FC form

$$\mathbf{U}_{,t} + \mathbf{F}^{(i)}{}_{,i} = 0 \tag{7}$$

$$\begin{aligned} \mathbf{U}(\mathbf{W}) &\equiv (D, S^1, S^2, S^3, \tau) \rightarrow \text{(conserved variables)} \\ \mathbf{W} &\equiv (\rho, v^i, \varepsilon) \rightarrow \text{(primitive variables)} \end{aligned}$$

$$\mathbf{F}^{(i)} = (Dv^i, S^j v^i + p\delta^{ji}, S^i - Dv^i) \;\; i, j = 1, 2, 3$$

$$\begin{aligned} D &= \rho\Gamma \equiv \text{rest-mass density} \\ S^j &= \rho h \Gamma^2 v^j \equiv \text{momentum density} \quad (j = 1, 2, 3) \\ \tau &= \rho h \Gamma^2 - p - \rho\Gamma \equiv \text{energy density} \end{aligned}$$

System (7) is hyperbolic for causal equations of state (Anile 1989), i.e., for those where the local sound speed, c_s, satisfies $c_s < 1$.

Equations of Special Relativistic Hydrodynamics (II)

▷ Eigenvalues for SRHD:

$$\lambda_0 = v^i, \quad \text{linearly degenerate (triple)}, \quad i = x, y, z$$

$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2} \left(v^i (1 - c_s^2) \pm c_s \sqrt{(1 - v^2)[1 - v^2 c_s^2 - v^i v^i (1 - c_s^2)]} \right) .$$

- Strong coupling with the x, y, z directions through $v^2 = (v^x)^2 + (v^y)^2 + (v^z)^2$.
- For the 1D case:

$$\lambda_\pm = \frac{v \pm c_s}{1 \pm v c_s} \rightarrow \begin{cases} 1 \ (v \rightarrow 1) \\ v \pm c_s \ (v, c_s \rightarrow 0) \end{cases}$$

▷ Hyperbolic systems of conservation laws admit discontinuous solutions (SHOCKS) ⇒ Satisfy Rankine-Hugoniot (RH) jump conditions across the hyper-surface containing the discontinuity of the space time, Σ, and are based on the continuity of the fluxes across shocks. In the case of SRHD these conditions (Taub 1948) read:

$$[\rho u^\mu] n_\mu = 0 \, , \tag{8}$$

$$[T^{\mu\nu}] n_\nu = 0 \, , \tag{9}$$

n_μ being the unit normal to Σ. Notation: $[F] = F_a - F_b$; F_a, $F_b \equiv$ values of F on the two sides of Σ.

Jump conditions across shocks

RH conditions (8), (9) can be written in terms of the conserved quantities and the invariant mass flux across the shock, j,

$$[v^x] = -\frac{j}{\Gamma_s}\left[\frac{1}{D}\right], \tag{10}$$

$$[p] = \frac{j}{\Gamma_s}\left[\frac{S^x}{D}\right], \tag{11}$$

$$j\left[\frac{S^{y,z}}{D}\right] = 0, \text{ or } j\,[hWv^{y,z}] = 0, \tag{12}$$

$$[v^x p] = \frac{j}{\Gamma_s}\left[\frac{\tau}{D}\right]. \tag{13}$$

Γ_s is the Lorentz factor associated to the shock.
If $j = 0$ (contact discontinuity) $\rightarrow$ continuous pressure and normal velocity, and an arbitrary jump in the tangential velocity.

Newtonian vs Relativistic Hydrodynamics

▷ Classical limit of the RHD equations ($c \to \infty$; $h \to 1$; $\Gamma \to 1$):

$$\mathbf{U} = (D, S^j, \tau) \to (\rho, \rho v^j, \frac{1}{2}\rho \mathrm{v}^2 + \rho\varepsilon)$$

$$\mathbf{F}^{(i)} = (Dv^i, S^j v^i + p\delta^{ji}, S^i - Dv^i) \to (\rho v^i, \rho v^j v^i + p\delta^{ji}, v^i(\frac{1}{2}\rho \mathrm{v}^2 + \rho\varepsilon + p))$$

$$i, j = 1, 2, 3$$

What is different from classical HD?:

- Equations are tightly coupled by Γ and h (larger non-linearity).
- No explicit relation between $\mathbf{W}$ and $\mathbf{U}$ (except for particular EOS).
- Coupling of the tangential components of v in the characteristic speeds (aberration).
- $v \to 1 \Rightarrow \lambda_0 \to \lambda_\pm \to 1 \Rightarrow$ total eigenfield degeneration $\Rightarrow$ very thin structures (*e.g.*, relativistic blast wave) $\Rightarrow$ potential source of numerical errors.
- Relativistic shocks can have larger jumps than classical ones.

Relativistic strong shock: $\frac{\rho_b}{\rho_a} \leq \frac{\gamma\Gamma_b + 1}{\gamma - 1} \to \infty$ if $v_b \to 1$

Newtonian strong shock: $\frac{\rho_b}{\rho_a} \leq \frac{\gamma + 1}{\gamma - 1} (\sim 4 - 7)$

Contents

Numerical integration of the RHD equations

• Late 60's - 80's → Artificial viscosity (AV): Application of von Neumann & Richtmyer (1950) to RHD.

standard finite difference techniques

+

viscous terms into the equations to damp spurious oscillations near discontinuities (*non-consistent AV* -AV not based on the stress-energy tensor of a viscous fluid-).

→ Basic idea: artificial dissipative mechanism that makes the shock transition smooth -extended over several numerical zones-.

Limitations:

Very diffusive, errors in shock velocity, test-dependent parameters, non conservative, not applicable to ultrarelativistic regime ($\Gamma \leq 2$),...

High resolution shock capturing (HRSC) methods

- 90's: The application of HRSC methods caused a revolution in numerical RHD because
  - The eqs. are written in conservation form
    - $\Rightarrow$ Convergence to the physically correct solution.
  - Exploit the hyperbolic character of the RHD equations (upwind).
    - $\Rightarrow$ automatically satisfies RH condition (shock capturing).
  - High resolution.
    - $\Rightarrow$ High order of accuracy in smooth regions of the flow.
    - $\Rightarrow$ Stable and sharp description of discontinuities.

Basics of HRSC methods:

1. Time evolution of zone averaged state vectors governed by the *numerical fluxes* evaluated at zone interfaces.

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \quad \Longleftrightarrow \quad \frac{d\mathbf{U}_j^n}{dt} = \underbrace{-\frac{1}{\Delta x}\left(\hat{\mathbf{F}}_{j+1/2} - \hat{\mathbf{F}}_{j-1/2}\right)}_{\text{spatial discretization}}$$

where $\Delta x = x_{j+1/2} - x_{j-1/2}$ and $\mathbf{U}_j^n$ is an approximation to $\mathbf{U}(x_j, t^n)$ (finite difference methods) or to the zone average (finite volume methods):

$$\bar{\mathbf{U}}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(t^n, x)dx\,, \tag{14}$$

$\hat{\mathbf{F}}_{j\pm1/2}$ are approximations to the time–averaged fluxes across the interfaces $x_{j\pm1/2}$:

$$\hat{\mathbf{F}}_{j\pm1/2} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(\overbrace{\mathbf{U}(x_{j\pm1/2}, t)}^{\text{sol. at the interface}})dt\,. \tag{15}$$

One always tries to represent as accurately as possible the numerical fluxes. Different ways of calculating $\hat{\mathbf{F}}_{j\pm1/2}$ yield different evolution schemes (Lax, Runge-Kutta, etc.).

▷ In the evaluation of $\hat{\mathbf{F}}_{j\pm1/2}$ we need to face the fact that discontinuities may develop or be present already in the initial data.

Basics of HRSC methods (ii):

To handle the discontinuities in the flow one can consider the following possibilities:

- **1st order accurate schemes**
  generally fine, but very inaccurate (e.g., excessive diffusion, with Lax method) or across discontinuities (e.g., upwind).
- **2nd order accurate schemes**
  generally introduce oscillations across discontinuities (not "monotonic" or TVD).
- **2nd order accurate schemes with artificial viscosity**
  mimic nature but not good in relativistic regimes (see previous slides).
- **Godunov Methods**
  good compromise between accuracy (2nd order with smooth data, 1st-order at discontinuities) but monotonic. Most importantly: discontinuities are a keystone of the algorithm.

Basics of HRSC methods (iii): Finite Volumes

Godunov methods are tightly related with finite-volume methods. For simplicity, assume a scalar equation discretized in a 1D uniform grid.

Finite-Volume Methods are based on subdividing the spatial domain into intervals ("finite volumes" or grid cells) and on keeping track of an approximation to the zone average

$$\bar{U}_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U(t^n, x)dx \tag{16}$$

over each of these volumes.

If $U(t, x)$ is smooth,

$$\bar{U}_j^n \simeq \frac{1}{\Delta x} U(t^n, x_i)\Delta x + \mathcal{O}(\Delta x^2) = U(t^n, x_i) + \mathcal{O}(\Delta x^2)$$

where $x_i = (x_{j+1/2} + x_{j-1/2})/2$. In other words, $\bar{U}_j^n$ agrees with $U(t, x)$ at the midpoint of the interval to $\mathcal{O}(\Delta x^2)$.

At each time step, we update these values using approximations to the flux through the endpoints of the intervals.

Basics of HRSC methods (iv): Finite Volumes

In terms of finite-volumes, it is easier to use important properties of the conservation laws in deriving numerical methods.

In particular, we can ensure that the numerical method is conservative in a way that mimics the true solution and this is important for correctly calculating shock waves.

The quantity

$$\sum_{i=1}^{N} \bar{U}_i^n \Delta x$$

approximates the integral of $U(t, x)$ over the entire interval $[a, b]$.

Using a method in conservative form, the discrete sum will change only due to the fluxes at the boundaries $x = a$ and $x = b$. In this way conservation (e.g. of mass) is guaranteed provided that the boundary conditions are properly imposed.

Basics of HRSC methods (v):

2. Numerical fluxes are obtained by means of *exact* or *approximate* RIEMANN SOLVERS.

Based on a simple, yet brilliant idea by Godunov (1959).

Basic idea: a piecewise constant description of hydrodynamical quantities produce a collection of local Riemann problems whose solution can be found exactly.

The solution at time t_{n+1} can be constructed by piecing together the Riemann solutions, provided that the time step is short enough that the waves from two adjacent Riemann problems have not started to interact yet.

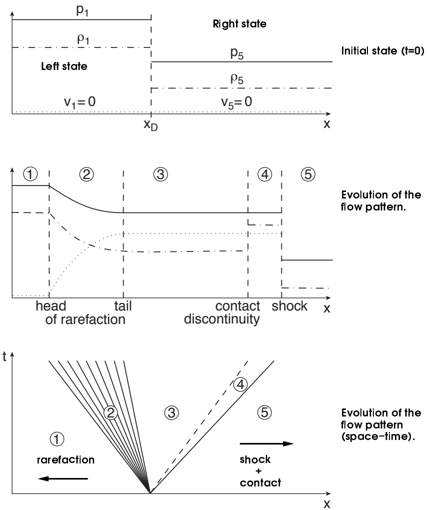
Basics of HRSC methods (vi):

Riemann problem (RP):
IVP with piecewise constant initial data:

$$u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases} \quad (17)$$

▷ **Solution:**
Set of CONSTANT states separated by centered *rarefactions* (selfsimilar expansions) or *shocks*.

Riemann solver:
Algorithm to evaluate the solution of a RP.



Exact Solution of the Riemann Problem in RHD. I.

(Thompson 1986; Martí & Müller, 1994; Pons, Martí & Müller, 2000)

- Riemann problem:

  IVP with initial discontinuous data L, R.

- Self-similar solution:

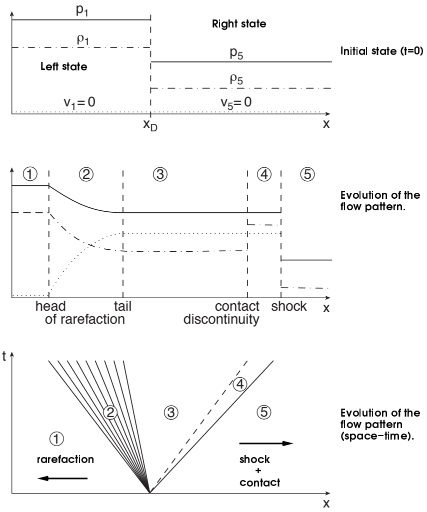
  $$LR \rightarrow \; L\, \mathcal{W}_{\leftarrow}\; L_{*}\; \mathcal{C}\; R_{*}\; \mathcal{W}_{\rightarrow}\; R$$

  $\mathcal{W}$ denotes a shock (discontinuos solution) or a rarefaction (selfsimilar expansion), and $\mathcal{C}$, a contact discontinuity

- The compressive character of shock waves allows us to discriminate between shocks ($\mathcal{S}$) and rarefaction waves ($\mathcal{R}$):

  $$\mathcal{W}_{\leftarrow\;(\rightarrow)} = \begin{cases} \mathcal{R}_{\leftarrow\;(\rightarrow)} & , \quad p_b \leq p_a \\ \mathcal{S}_{\leftarrow\;(\rightarrow)} & , \quad p_b > p_a \end{cases}$$

  where p is the pressure and subscripts a and b denote quantities ahead and behind the wave.



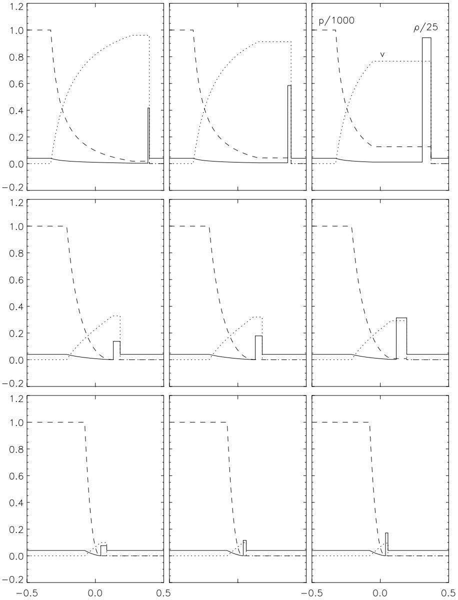
Exact Solution of the Riemann Problem in RHD. II.

Intrinsic relativistic effects

⇕

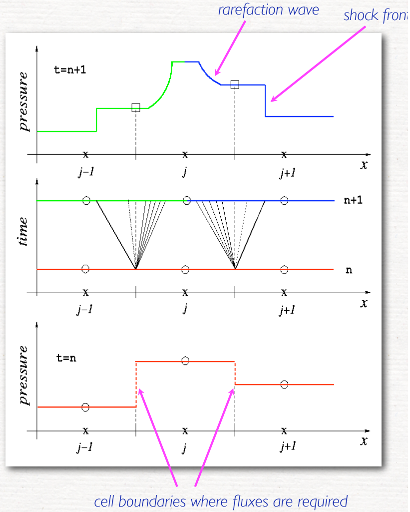
Coupling of tangential speeds

- In RHD, all the components of the flow velocity are coupled, through the Lorentz factor, in the solution of the Riemann problem.
- In addition, the specific enthalpy also couples with the tangential velocities, which becomes important in the thermodynamically ultrarelativistic regime
- Analytical pressure, density and flow velocity profiles at $t = 0.4$ for the relativistic Riemann problem with initial data $p_L = 10^3$, $\rho_L = 1.0$, $v_L^x = 0.0$; $p_R = 10^{-2}$, $\rho_R = 1.0$ and $v_R^x = 0.0$, varying the values of the tangential velocities. From left to right, $v_R^t = 0, 0.9, 0.99$ and from top to bottom $v_L^t = 0, 0.9, 0.99$. $\gamma = 5/3$.



In HRSC Godunov methods the numerical flux is given by

$$\hat{\mathbf{F}}_{j\pm 1/2} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(\tilde{\mathbf{U}}(t, x_{j\pm 1/2}))dt\,,$$



$\Rightarrow$ $\tilde{\mathbf{U}}(x_{j\pm 1/2}, t)$ is calculated solving RPs at every zone interface with initial data

$$\tilde{\mathbf{U}}(t^n, x_{j\pm 1/2}) = \begin{cases} U_L(t^n, x) & x < x_{j\pm 1/2} \\ U_R(t^n, x) & x > x_{j\pm 1/2} \end{cases}$$

Solution at the time $n+1$ of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$.

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions.

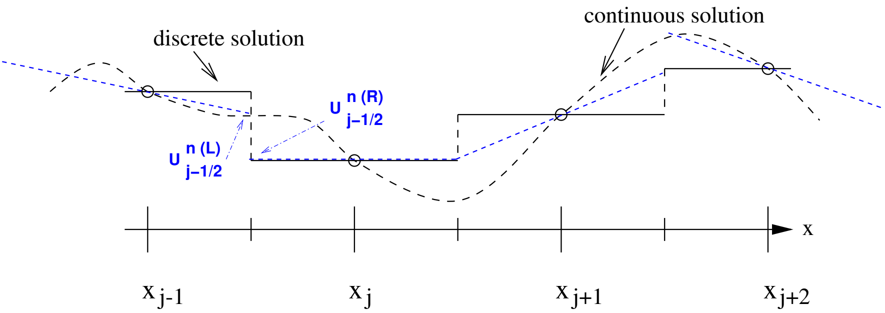
Initial data at the time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$.

Credit: Rezzolla (2008)

Basics of HRSC methods (vii):

3. High–order of accuracy ⇐ conservative monotonic* polynomial functions to interpolate the approximate solution within zones.

* monotonic functions lead to the decrease of the total variation (total–variation–diminishing schemes, TVD; Harten 1984), ensuring stability.



- 1^{st} order → piecewise constant functions (RP: $\mathbf{U}^{\rm L}_{j-1/2} = \mathbf{U}^n_{j-1}$, $\mathbf{U}^{\rm R}_{j-1/2} = \mathbf{U}^n_j$)
- 2^{nd} order → piecewise linear functions (MUSCL, van Leer 1979)
- $\geq 3^{rd}$ order → piecewise quadratic functions (PPM, Colella & Woodward 1984; PHM, Marquina 1994), WENO (Liu et al. 1994), MP (Suresh & Huynh 1997), etc..

Basics of HRSC methods (viii):

4. Transformation from primitive to conserved variables

Once the solution in terms of the conserved variables $\mathbf{U} = (D, S_j, \tau)^T$ has been obtained, it is necessary to return to the primitive variables after inverting numerically the set of equations

$$\begin{array}{rcll} D & = & \rho\Gamma\,, & \rho\,, \\ S_j & = & \rho h \Gamma^2 v_j\,, \quad (j = 1,2,3) \quad \Longrightarrow & v_j\,, \\ \tau & = & \rho h \Gamma^2 - p - \rho\Gamma\,, & \epsilon\,. \end{array}$$

Note: this conversion cannot be done analytically and requires the solution of a set of coupled eqs. Note that this numerical procedure is specific of RHD (GRHD, RMHD, GRMHD). This root-finding operation is very expensive computationally.

This series of operations is repeated at each grid point and for each time level...

HRSC ($\neq$ Riemann solver based)

Flux Corrected Transport (FCT)
Higher accuracy is obtained by adding an anti–diffusive flux term to the 1^{st}–order numerical flux (Boris & Book 1973). The interpolation algorithms have to preserve the TV–stability of the scheme.

Symmetric Total Variation Diminishing (TVD) schemes + nonlinear numerical dissipation

- As TVD schemes verify: $\mathrm{TV}(u^{n+1}) \leq \mathrm{TV}(u^n), \;\forall n$
- written in conservation form,
- local conservative dissipation terms $\Rightarrow$ NOT based on RS.
- GRMHD: Koide *et al.* (1996, 1997); Nishikawa *et al.* (1998).

Relativistic beam scheme

▷ RHD eqs. are solved as the limit of the Boltzmann equation (Yang *et al.* 1997).

- The Jüttner distribution function is approximated by Dirac delta functions (beams of particles), which reproduce the appropriate moments of the distribution function.
- The integration scheme can be cast in the form of an upwind conservation scheme -and extended to higher–order (TVD2, ENO2, ENO3)-.

Other Methods

Van Putten's approach

Van Putten (1993) solves the eqs. of (ideal) SRMHD formulating Maxwell's equations as a hyperbolic system in divergence form. $\mathbf{U}$ and $\mathbf{F}$ are decomposed into a spatially constant mean and a spatially dependent variational parts. Then,

▷ The SRMHD eqs. become a system of evolution equations for integrated (continuous) quantities ⇒ standard methods can be used to integrate the eqs.
Applications: SRMHD jets ($\Gamma < 4.25$) -van Putten 1993b, 1996-.

Relativistic Smoothed Particle Hydrodynamics (RSPH)

SPH (Lucy 1977) represents a fluid by a Monte Carlo sampling of its mass elements. The motion and thermodynamics of these mass elements is governed by the HD eqs.
⇒ Free-Lagrange method (no computational grid basis).

Extension to SRHD: Monaghan (1985). Other codes: Lahy (1989, SRHD); Kheyfets, Miller & Zurek (1990, GRHD); Mann (1991, 1993, SRHD); Laguna, Miller & Zurek (1993, GRHD); Chow & Monaghan (1997, SRHD); Siegler, & Riffert (1999, GRHD).

▷ Monaghan (1997) and Chow & Monaghan (1997) incorporate concepts from RSs into SPH. Siegler, & Riffert (1999) include consistent AV.

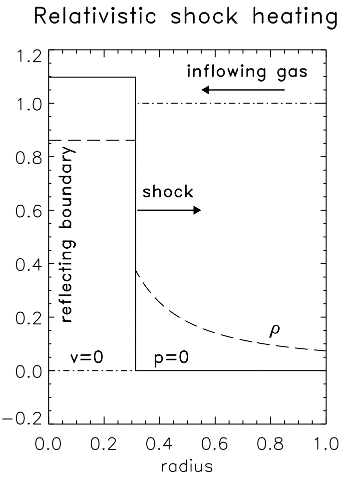
Contents

Validation: Relativistic shock reflection

Shock heating of a cold fluid in planar, cylindrical, or spherical geometry has been used as a test case for hydrodynamic codes, because:

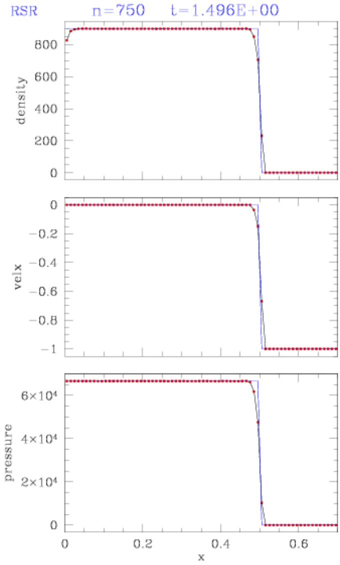
- Has an analytical solution ([26] planar sym., [183] cylindrical and spherical sym.)
- Involves the propagation of a strong relativistic shock.
- Is a simplified model of a common situation in physics: implosion, collapse, collision of two streams of plasma,...

Initial values		
	Left	Right
p	0.00	0.00
ρ	1.00	1.00
v	$-v_1$	v_1
$\sigma_{\rm shock}$	$\frac{\gamma+1}{\gamma-1} + \frac{\gamma(\Gamma_1-1)}{\gamma-1}$	
$\gamma = 4/3$	$\sigma_{\rm shock}^{\rm Newton} = 7$	



Martí & Müller (2003) Liv. Revs. Rel.

Validation: Relativistic shock reflection



Martí & Müller (2003) Liv. Revs. Rel.

- Specific energy of the shocked matter:

$$\epsilon_2 = \Gamma_1 - 1$$

- Shock velocity:

$$V_s = \frac{(\gamma - 1)\Gamma_1 |v_1|}{\Gamma_1 + 1}$$

- Self-similar density distribution in the (pressure-less) upstream state:

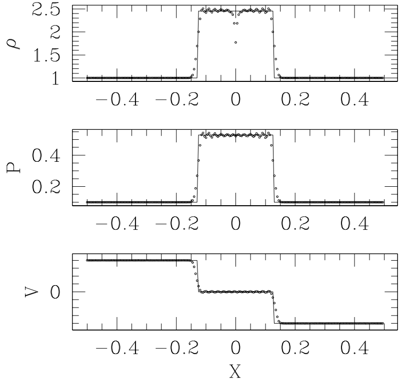
$$\rho(t,r) = \left(1 + \frac{|v_1| t}{r}\right)^{\alpha} \rho_0$$

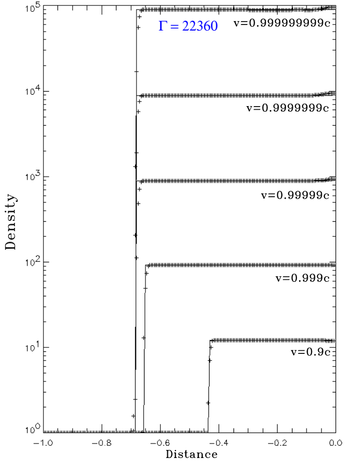
$\alpha = 0, 1, 2$ for planar, cylindrical or spherical geometry.

Validation: Relativistic shock reflection

References	α	Method	$\Gamma_{\max}$	$\sigma_{\text{error}}(\%)$
Centrella and Wilson (1984)	0	AV-mono	2.29	$\simeq 10$
Hawley et al. (1984)	0	AV-mono	4.12	$\simeq 10$
Norman and Winkler (1986)	0	cAV-implicit	10.0	0.01
McAbee et al. (1989)	0	AV-mono	10.0	2.6
Martí et al. (1991)	0	Roe type-l	23	0.2
Marquina et al. (1992)	0	LCA-phm	70	0.1
Eulderink (1993)	0	Roe–Eulderink	625	$\lesssim 0.1$
Schneider et al. (1993)	0	RHLLE	106	0.2
	0	SHASTA-c	106	0.5
Dolezal and Wong (1995)	0	LCA-eno	7.0^5	$\lesssim 0.1$
Martí and Müller (1996)	0	rPPM	224	0.03
Falle and Komissarov (1996)	0	Falle–Komissarov	224 $\lesssim 0.1$	
Romero et al. (1996)	2	Roe type-l	2236	2.2
Martí et al. (1997)	1	MFF-ppm	70	1.0
Chow and Monaghan (1997)	0	SPH-RS-c	70	0.2
Wen et al. (1997)	2	rGlimm	224	10^{-9}
Donat et al. (1998)	0	MFF-eno	224	$\lesssim 0.1$
Aloy et al. (1999)	0	MFF-ppm	2.4×10^5	3.57
Sieglert and Riffert (1999)	0	SPH-cAV-c	1000	$\lesssim 0.1$
Del Zanna and Bucciantini (2002)	0	sCENO	224	2.3
Anninos and Fragile (2002)	0	cAV-mono	4.12	13.3
	0	NOCD	2.4×10^5	0.1

Validation: Relativistic shock reflection





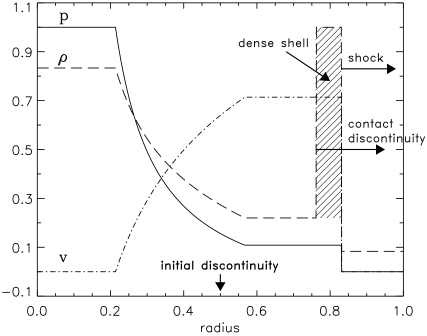
Shibata *et al.* (1999) PRD 60, 104052

Aloy *et al.* (1999), ApJS, 122, 151

Validation: Relativistic Blast Waves

RPs with large initial pressure jumps produce blast waves with dense shells of material propagating at relativistic speeds. For appropriate initial conditions, both $v_{\rm shock}$ and $v_{\rm shell}$ approach c producing very narrow structures. The accurate description of these thin, relativistic shells involving large density contrasts is a challenge for any numerical code. Some particular blast wave problems have become **standard numerical tests**.

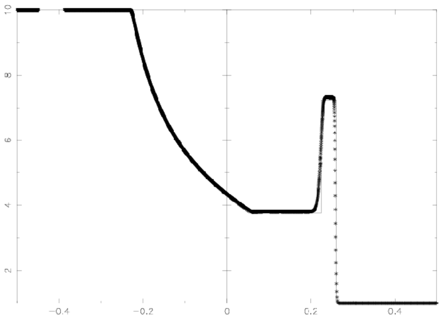
Generation and propagation of relativistic blast waves



Weak blast wave Initial values		
	Left	Right
p	13.33	0.00
ρ	10.00	1.00
v	0.00	0.00
$v_{\rm shell}$	0.72	
$w_{\rm shell}$	0.11t	
$v_{\rm shock}$	0.83	
$\sigma_{\rm shock}$	5.07	
$\gamma = 5/3$	$\sigma_{\rm shock}^{\rm Newton} = 4$	

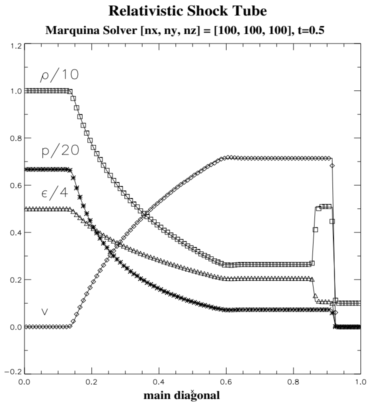
Martí & Müller (2003) Liv. Revs. Rel.

Validation: Relativistic Blast Waves



Density distribution for the weak relativistic blast wave (t=0.314) using 5500 SPH particles (Muir 2002; PhD Thesis).

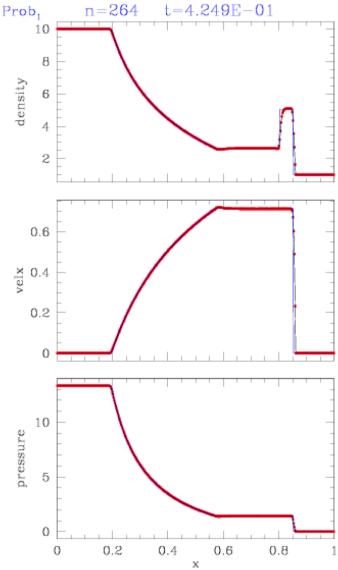
Credit: Martí & Müller (2003) Liv. Revs. Rel.

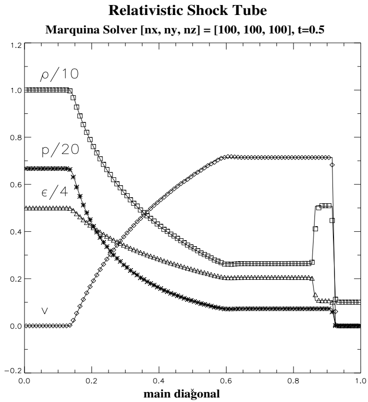


Aloy *et al.* (1999), ApJS, 122, 151

3D version of the weak blast wave using 128^3 uniform zones.

Validation: Relativistic Blast Waves



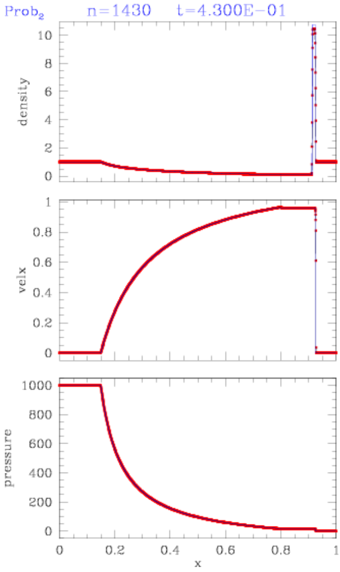


Aloy *et al.* (1999), ApJS, 122, 151

3D version of the weak blast wave using 128^3 uniform zones.

Martí & Müller (2003) Liv. Revs. Rel.

Validation: Relativistic Blast Waves



Strong blast wave Initial values		
	Left	Right
p	1000.00	0.01
ρ	1.00	1.00
v	0.00	0.00
v_{shell}	0.960	
w_{shell}	0.026t	
v_{shock}	0.986	
σ_{shock}	10.75	
$\gamma = 5/3$	$\sigma_{\text{shock}}^{\text{Newton}} = 4$	

Martí & Müller (2003) Liv. Revs. Rel.

Validation: Relativistic Blast Waves

Shock compression ratios for runs with 400 zones at $t \simeq 0.4$

References	Method	σ/σ_{ex}
Norman and Winkler (1986)	cAV-implicit	1.00 [(a)]
Dubal (1991)	FCT-lw	0.80
Martí et al. (1991)	Roe type-l	0.53
Marquina et al. (1992)	LCA-phm	0.64
Martí and Müller (1996)	rPPM	0.68
Falle and Komissarov (1996)	FK	0.47
Wen et al. (1997)	rGlimm	1.00
Chow and Monaghan (1997)	SPH-RS-c	1.16 [(b)]
Donat et al. (1998)	MFF-phm	0.60
Del Zanna and Bucciantini (2002)	sCENO	0.69
Anninos and Fragile (2002)	cAV-mono	1.40 [(c)]
	NOCD	0.67 [(c)]

(a) Adaptative grid

(b) At $t = 0.15$

(c) With 800 zones

Martí & Müller (2003) Liv. Revs. Rel.

Strong blast wave

Initial values

	Left	Right
p	1000.00	0.01
ρ	1.00	1.00
v	0.00	0.00
$v_{\rm shell}$	0.960	
$w_{\rm shell}$	$0.026t$	
$v_{\rm shock}$	0.986	
$\sigma_{\rm shock}$	10.75	
$\gamma = 5/3$	$\sigma_{\rm shock}^{\rm Newton} = 4$	

Convergence under grid refinement:

How do we know that the method/code works well if there is no analytic solution to confront with?

$$\underbrace{||E_{\Delta x}||}_{\text{global error}} = \Delta x \sum_j |\bar{\mathbf{U}}_j^n - \mathbf{U}_j^n| \to 0 \quad \text{if} \quad \Delta x \to 0$$

To guarantee convergence, stability is required (Lax equivalence theorem) $\rightarrow$ total–variation stability (powerful theoretical results only for scalar conservation laws). The total variation of a solution at $t = t^n$, TV($\mathbf{U}^n$), is defined as

$$\mathrm{TV}(\mathbf{U}^n) = \sum_{j=0}^{+\infty} |\mathbf{U}_{j+1}^n - \mathbf{U}_j^n| \, . \tag{18}$$

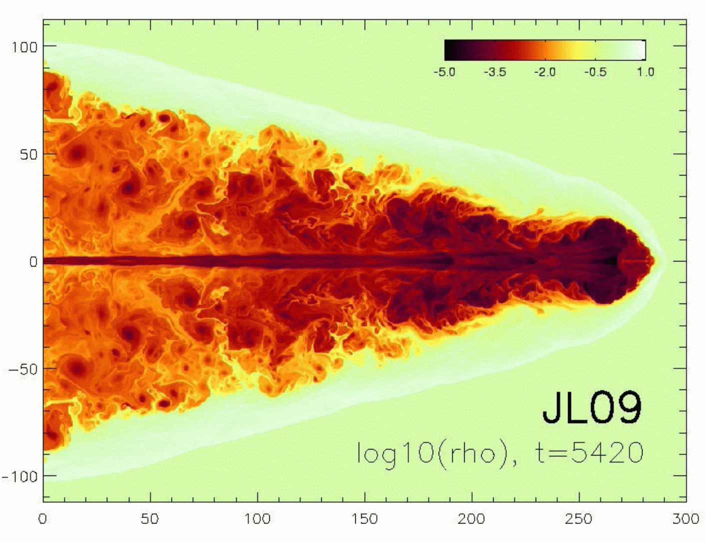
A numerical scheme is said to be TV–stable, if TV($\mathbf{U}^n$) is bounded for all Δt at any time for each initial data.

Modern research has focussed on the development of high–order, accurate methods in conservation form (usually by using the integral version of the eqs.) + consistent numerical fluxes + TV–stability.

Contents

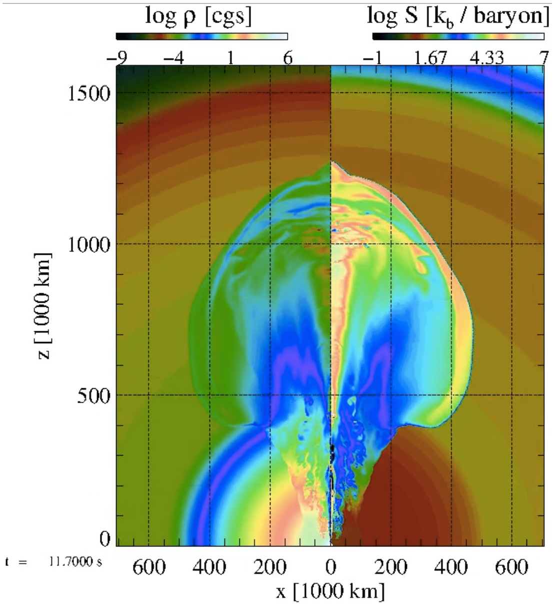
Applications (a partisan view)

Extragalactic jets



Scheck, Aloy, Martí, Gómez & Müller (2002)

Applications (a partisan view)



GRB progenitors

Obergaulinger & Aloy (2015), in prep.

Contents

Summary

- The solution of the hydrodynamics equations requires special care because of their nonlinear
- Even smooth initial data tends to steepen and shock; in addition any discretization leads to small discontinuities
- Using a flux-conservative formulation is essential if modelling discontinuities
- HRSC methods are particularly suited to study discontinuities since they are based on them via Riemann problems